Some properties of the Fejér kernel

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Abstract

This paper investigates key properties of the Fejér kernel, a fundamental concept in harmonic analysis and Fourier series. The Fejér kernel, defined as the arithmetic mean of the first n partial sums of the Fourier series, is known for its role in the Cesàro summation method. We prove several important properties of the Fejér kernel.

1 Properties

Define

$$F_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x), \quad n \in \mathcal{N}. \tag{1}$$

as the Fejér kernel and

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}, \quad n = 0, 1, 2, \dots$$

as the Dirichlet kernel. As such, the following results hold

(a) We have

$$F_n(x) = \begin{cases} \dfrac{1}{n}\left(\dfrac{\sin(nx/2)}{\sin(x/2)}\right)^2, & \text{if } x/2\pi \notin \mathcal{Z}; \\ n, & \text{if } x/2\pi \in \mathcal{Z}. \end{cases}$$

(b) $(2\pi)^{-1}\int_{-\pi}^{\pi} F_n(t)dt = 1$.
(c) $F_n(-x) = F_n(x)$ for each $x \in \mathcal{R}$.
(d) For each $\delta \in (0, \pi)$, $\lim_{n\to\infty} \sup\{F_n(x) : \delta \le |x| \le \pi\} = 0$.
(e) If $f \in \mathcal{L}^1_{2\pi}$, then $A_n(f)(x) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(t)F_n(x-t)dt$ for each $x \in \mathcal{R}$.

2 Proofs

Proof. (b) Substituting the definition of the Dirichlet kernel, into (1), we get

$$F_n(t) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k} e^{imt}.$$

Hence, integrating $F_n(t)$ from $-\pi$ to π,

$$\begin{aligned}(2\pi)^{-1}\int_{-\pi}^{\pi} F_n(t)dt &= (2\pi)^{-1}\int_{-\pi}^{\pi}\left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k} e^{imt}\right)dt \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k}\left((2\pi)^{-1}\int_{-\pi}^{\pi} e^{imt}dt\right).\end{aligned} \tag{2}$$

For $m > 0$, where m is an integer, we have

$$\begin{aligned}(2\pi)^{-1}\int_{-\pi}^{\pi} e^{imt}dt &= (2\pi)^{-1}\int_{-\pi}^{\pi}\left[\cos(mt) + i\sin(mt)\right]dt \\ &= (2\pi)^{-1}\left[\int_{-\pi}^{\pi}\cos(mt)dt + i\int_{-\pi}^{\pi}\sin(mt)dt\right] \\ &= (2\pi)^{-1}\left[\frac{\sin(mt)}{m}\Big|_{-\pi}^{\pi} - i\frac{\cos(mt)}{m}\Big|_{-\pi}^{\pi}\right] \\ &= (2\pi)^{-1}[0 - i(0)] \\ &= 0.\end{aligned}$$

For $m = 0$, we have

$$\begin{aligned}(2\pi)^{-1}\int_{-\pi}^{\pi} e^{i0t}dt &= (2\pi)^{-1}\int_{-\pi}^{\pi} 1dt \\ &= (2\pi)^{-1}[\pi - (-\pi)] \\ &= (2\pi)^{-1}(2\pi) \\ &= 1.\end{aligned}$$

Hence, (2) becomes

$$(2\pi)^{-1}\int_{-\pi}^{\pi} F_n(t)dt = \frac{1}{n}\sum_{k=0}^{n-1} 1 = \frac{1}{n}\sum_{k=1}^{n} 1 = \frac{1}{n}n = 1.$$

□

Proof. (c) First, for $x \in \mathcal{R}$ such that $x/2\pi \notin \mathcal{Z}$, then by part (a) we have

$$F_n(x) = \frac{1}{n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2.$$

By the negative angle identity, $\sin(-x) = -\sin(x)$, so

$$\begin{aligned}F_n(-x) = \frac{1}{n}\left(\frac{\sin(-nx/2)}{\sin(-x/2)}\right)^2 &= \frac{1}{n}\left(\frac{-\sin(nx/2)}{-\sin(x/2)}\right)^2 \\ &= \frac{1}{n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2 \\ &= F_n(x).\end{aligned}$$

Second, for $x \in \mathcal{R}$ such that $x/2\pi \in \mathcal{Z}$, then $x = 2m\pi$ and $-x = -2m\pi$ for some integer m. Hence, by part (a)

$$F_n(x) = n$$

and

$$F_n(-x) = n.$$

Therefore, in either case, $F_n(x) = F_n(-x)$ for each $x \in \mathcal{R}$. □

Proof. (d) Let $\delta \in (0, \pi)$. Since $\delta \leq |x| \leq \pi$, we have $0 < \delta < |x|$, so $\sin^2 x > \sin^2 \delta$. From part (a), we have

$$F_n(x) = \frac{1}{n}\left(\frac{\sin^2(nx/2)}{\sin^2(x/2)}\right)$$

since for all $x \in [\delta, \pi]$, where $\delta \in (0, \pi)$, we have $x/2\pi \notin \mathcal{Z}$. Hence,

$$\begin{aligned} F_n(x) = \frac{1}{n}\left(\frac{\sin^2(nx/2)}{\sin^2(x/2)}\right) &< \frac{1}{n}\left(\frac{\sin^2(nx/2)}{\sin^2(\delta/2)}\right) \\ &= \frac{1}{n\sin^2(\delta/2)}\sin^2(nx/2) \end{aligned}$$

Taking the supremum of both sides, for $\delta \leq |x| \leq \pi$, we get

$$\begin{aligned} \sup_{\delta \leq |x| \leq \pi} F_n(x) &< \frac{1}{n\sin^2(\delta/2)} \sup_{\delta \leq |x| \leq \pi} \sin^2(nx/2) \\ &\leq \frac{1}{n\sin^2(\delta/2)} \cdot 1 \end{aligned} \tag{3}$$

since $\sin^2(nx/2)$ alternates between 0 and 1. Taking the limit as $n \to \infty$ of both sides of (3), we get

$$\begin{aligned} \lim_{n\to\infty} \sup_{\delta \leq |x| \leq \pi} F_n(x) &\leq \lim_{n\to\infty} \frac{1}{n\sin^2(\delta/2)} \\ &= 0. \end{aligned} \tag{4}$$

Also, by part (a), $F_n(x) \geq 0$ for all x. As such, $\lim_{n\to\infty} \sup_{\delta \leq |x| \leq \pi} F_n(x) \geq 0$. Combining the previous sentence with (4), we get for each $\delta \in (0, \pi)$, $\lim_{n\to\infty} \sup\{F_n(x) : \delta \leq |x| \leq \pi\} = 0$, as desired. □

Proof. (e) For $f \in \mathcal{L}^1_{2\pi}$, we have

$$A_n(f)(x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(f)(x). \tag{5}$$

Let $S_n(f)$ denote the nth partial sum of the Fourier series of f:

$$S_n(f)(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx}. \tag{6}$$

For $f \in \mathcal{L}^1([-\pi, \pi])$, the function $\hat{f} : \mathcal{Z} \to \mathbb{C}$ defined by

$$\hat{f}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-imt} dt. \tag{7}$$

is called the Fourier transform of f. Substituting (6) and (7) into (5), we get

$$\begin{aligned}
A_n(f)(x) &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-k}^{k} \hat{f}(m) e^{imx} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-k}^{k} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-imt} dt \right] e^{imx} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-k}^{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-imt} e^{imx} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=-k}^{k} e^{im(x-t)} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n} \sum_{k=0}^{n-1} D_k(x-t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) F_n(x-t) dt
\end{aligned}$$

for all $x \in \mathcal{R}$. □